

Trigonometric Functions

- If in a circle of radius r , an arc of length l subtends an angle of θ radians, then $l = r\theta$.
- Radian measure $= \frac{\pi}{180} \times \text{Degree measure}$
- Degree measure $= \frac{180}{\pi} \times \text{Radian measure}$
- A degree is divided into 60 minutes and a minute is divided into 60 seconds. One sixtieth of a degree is called a minute, written as $1'$, and one sixtieth of a minute is called a second, written as $1''$.

Thus, $1^\circ = 60'$ and $1' = 60''$

- Signs of trigonometric functions in different quadrants:**

Trigonometric function	Quadrant I	Quadrant II	Quadrant III	Quadrant IV
$\sin x$	+ ve (Increases from 0 to 1)	+ ve (Decreases from 1 to 0)	-ve (Decreases from 0 to -1)	-ve (Increases from -1 to 0)
$\cos x$	+ ve (Decreases from 1 to 0)	-ve (Decreases from 0 to -1)	-ve (Increases from -1 to 0)	+ ve (Increases from 0 to 1)
$\tan x$	+ ve (Increases from 0 to ∞)	-ve (Increases from $-\infty$ to 0)	+ ve (Increases from 0 to ∞)	-ve (Increases from $-\infty$ to 0)
$\cot x$	+ ve (Decreases from ∞ to 0)	-ve (Decreases from 0 to $-\infty$)	+ ve (Decreases from ∞ to 0)	-ve (Decreases from 0 to $-\infty$)
$\sec x$	+ ve (Increases from 1 to ∞)	-ve (Increases from $-\infty$ to -1)	-ve (Decreases from -1 to $-\infty$)	+ ve (Decreases from ∞ to 1)
$\operatorname{cosec} x$	+ ve (Decreases from ∞ to 1)	+ ve (Increases from 1 to ∞)	-ve (Increases from $-\infty$ to -1)	-ve (Decreases from -1 to $-\infty$)

Example 1:

If $\sin \theta = -\frac{1}{\sqrt{3}}$, where $\pi < \theta < \frac{3\pi}{2}$, then find the value of $3 \tan \theta - \sqrt{3} \sec \theta$.

Solution: Since θ lies in the third quadrant, therefore $\tan \theta$ is positive and $\cos \theta$ (or $\sec \theta$) is negative.

$$\cos^2 \theta + \sin^2 \theta = 1$$

$$\Rightarrow \cos \theta = \pm \sqrt{1 - \sin^2 \theta}$$

$$\Rightarrow \cos \theta = \pm \sqrt{1 - \left(-\frac{1}{\sqrt{3}}\right)^2} = \pm \sqrt{1 - \frac{1}{3}} = \pm \sqrt{\frac{2}{3}}$$

$$\therefore \cos \theta = -\sqrt{\frac{2}{3}}$$

$$\Rightarrow \sec \theta = -\sqrt{\frac{3}{2}}$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{-\frac{1}{\sqrt{3}}}{-\sqrt{\frac{2}{3}}} = \frac{1}{\sqrt{2}}$$

$$\therefore 3 \tan \theta - \sqrt{3} \sec \theta = 3 \times \frac{1}{\sqrt{2}} - \sqrt{3} \times \left(-\sqrt{\frac{3}{2}}\right) = \frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}} = 3\sqrt{2}$$

Example 2: Find the value of $\cos 390^\circ \cos 510^\circ + \sin 390^\circ \cos (-660^\circ)$.

Solution:

$$\cos 390^\circ = \cos (2 \times 180^\circ + 30^\circ) = \cos 30^\circ = \frac{\sqrt{3}}{2}$$

$$\cos 510^\circ = \cos (3 \times 180^\circ - 30^\circ) = -\cos 30^\circ = -\frac{\sqrt{3}}{2}$$

$$\sin 390^\circ = \sin (2 \times 180^\circ + 30^\circ) = \sin 30^\circ = \frac{1}{2}$$

$$\cos (-660^\circ) = \cos 660^\circ = \cos (4 \times 180^\circ - 60^\circ) = \cos 60^\circ = \frac{1}{2}$$

$$\therefore \cos 390^\circ \cos 510^\circ + \sin 390^\circ \cos (-660^\circ)$$

$$= \frac{\sqrt{3}}{2} \times \left(-\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{2}\right) \times \left(\frac{1}{2}\right)$$

$$= -\frac{3}{4} + \frac{1}{4}$$

$$= -\frac{2}{4}$$

$$= -\frac{1}{2}$$

- Domain and Range of trigonometric functions:

Trigonometric function	Domain	Range
$\sin x$	$\mathbf{R}$	$[-1, 1]$
$\cos x$	$\mathbf{R}$	$[-1, 1]$
$\tan x$	$\mathbf{R} - \left\{ x : x = \frac{(2n+1)\pi}{2}, n \in \mathbf{Z} \right\}$	$\mathbf{R}$
$\cot x$	$\mathbf{R} - \{ x : x = n\pi, n \in \mathbf{Z} \}$	$\mathbf{R}$
$\sec x$	$\mathbf{R} - \left\{ x : x = \frac{(2n+1)\pi}{2}, n \in \mathbf{Z} \right\}$	$\mathbf{R} - [-1, 1]$
$\operatorname{cosec} x$	$\mathbf{R} - \{ x : x = n\pi, n \in \mathbf{Z} \}$	$\mathbf{R} - [-1, 1]$

- Trigonometric identities and formulas:

$$\operatorname{cosec} x = \frac{1}{\sin x}$$

$$\sec x = \frac{1}{\cos x}$$

$$\tan x = \frac{\sin x}{\cos x}$$

$$\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$$

$$\cos^2 x + \sin^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$1 + \cot^2 x = \operatorname{cosec}^2 x$$

$$\cos(2n\pi + x) = \cos x, n \in \mathbf{Z}$$

$$\sin(2n\pi + x) = \sin x, n \in \mathbf{Z}$$

$$\sin(-x) = -\sin x$$

$$\cos(-x) = \cos x$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x$$

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x$$

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$

$$\cos\left(\frac{\pi}{2} + x\right) = -\sin x$$

$$\sin\left(\frac{\pi}{2} + x\right) = \cos x$$

$$\cos(\pi - x) = -\cos x$$

$$\sin(\pi - x) = \sin x$$

$$\cos(\pi + x) = -\cos x$$

$$\sin(\pi + x) = -\sin x$$

$$\cos(2\pi - x) = \cos x$$

$$\sin(2\pi - x) = -\sin x$$

- If none of the angles x, y and $(x \pm y)$ is an odd multiple of $\frac{\pi}{2}$, then

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}, \text{ and } \tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

- If none of the angles x, y and $(x \pm y)$ is a multiple of π , then $\cot(x + y) = \frac{\cot x \cot y - 1}{\cot y + \cot x}$,
and $\cot(x - y) = \frac{\cot x \cot y + 1}{\cot y - \cot x}$

$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \frac{1 - \tan^2 x}{1 + \tan^2 x}$$

$$\cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} = 2\cos^2 \frac{x}{2} - 1 = 1 - 2\sin^2 \frac{x}{2} = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$$

- In particular,

$$\sin 2x = 2\sin x \cos x = \frac{2 \tan x}{1 + \tan^2 x}$$

$$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2} = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$$

In particular,

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

In particular,

• **General solutions of some trigonometric equations:**

- $\sin x = 0 \Rightarrow x = n\pi$, where $n \in \mathbf{Z}$
- $\cos x = 0 \Rightarrow x = (2n + 1)\frac{\pi}{2}$, where $n \in \mathbf{Z}$
- $\sin x = \sin y \Rightarrow x = n\pi + (-1)^n y$, where $n \in \mathbf{Z}$
- $\cos x = \cos y \Rightarrow x = 2n\pi \pm y$, where $n \in \mathbf{Z}$
- $\tan x = \tan y \Rightarrow x = n\pi + y$, where $n \in \mathbf{Z}$

Example 1: Solve $\cot x \cos^2 x = 2 \cot x$

Solution:

$$\cot x \cos^2 x = 2 \cot x$$

$$\Rightarrow \cot x \cos^2 x - 2 \cot x = 0$$

$$\Rightarrow \cot x (\cos^2 x - 2) = 0$$

$$\Rightarrow \cot x = 0 \text{ or } \cos^2 x = 2$$

$$\Rightarrow \frac{\cos x}{\sin x} = 0 \text{ or } \cos x = \pm \sqrt{2}$$

$$\Rightarrow \cos x = 0 \text{ or } \cos x = \pm \sqrt{2}$$

$$\text{Now, } \cos x = 0 \Rightarrow x = (2n + 1)\frac{\pi}{2}, \text{ where } n \in \mathbf{Z}$$

$$\text{and } \cos x = \pm \sqrt{2}$$

But this is not possible as $-1 \leq \cos x \leq 1$

Thus, the solution of the given trigonometric equation is $x = (2n + 1)\frac{\pi}{2}$ where $n \in \mathbf{Z}$.

Example 2: Solve $\sin 2x + \sin 4x + \sin 6x = 0$.

Solution:

$$\sin 4x + (\sin 2x + \sin 6x) = 0$$

$$\Rightarrow \sin 4x + 2 \sin\left(\frac{2x+6x}{2}\right) \cos\left(\frac{2x-6x}{2}\right) = 0$$

$$\Rightarrow \sin 4x + 2 \sin 4x \cos 2x = 0$$

$$\Rightarrow \sin 4x(1 + 2 \cos 2x) = 0$$

$$\Rightarrow \sin 4x = 0 \text{ or } 1 + 2 \cos 2x = 0$$

$$\Rightarrow \sin 4x = 0 \text{ or } \cos 2x = -\frac{1}{2}$$

$$\sin 4x = 0$$

$$\Rightarrow 4x = n\pi, n \in \mathbb{Z}$$

$$\Rightarrow x = \frac{n\pi}{4}, n \in \mathbb{Z}$$

$$\cos 2x = -\frac{1}{2}$$

$$\Rightarrow \cos 2x = \cos \frac{2\pi}{3}$$

$$\Rightarrow 2x = 2m\pi \pm \frac{2\pi}{3}, m \in \mathbb{Z}$$

$$\Rightarrow x = m\pi \pm \frac{\pi}{3}, m \in \mathbb{Z}$$

$$\text{Thus, } x = \frac{n\pi}{4} \text{ or } x = m\pi \pm \frac{\pi}{3}, \text{ where } m, n \in \mathbb{Z}$$